

# RICCI FLOW WITH HYPERBOLIC WARPED PRODUCT METRICS

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ABSTRACT. In this short note, we show that the negative curvature is preserved in the deformation of hyperbolic warped product metrics under Ricci flow. It is also showed that the flow converges to a flat metric as time going to infinity.

## 1. INTRODUCTION

In this short note, we show by an example that the negative curvature is preserved in the deformation of hyperbolic warped product metrics under Ricci flow. It is also showed that the Ricci flow converges to a flat metric as time going to infinity. Given a time interval  $I = (a, b)$ . Recall that the Ricci flow on a non-compact manifold  $M^{n+1}$  is defined by a one parameter family of metrics  $\{g(t); t \in I\}$  satisfying

$$(1.1) \quad \partial_t g(t) = -2Rc(g(t)), \quad t \in I$$

where  $Rc(g(t))$  is the Ricci tensor of the metric  $g(t)$ . We denote by  $R(t)$  the scalar curvature of  $g(t)$ . Then the scalar curvature function  $R(t)$  satisfies the following evolution equation

$$(1.2) \quad \partial_t R = \Delta R + 2|Rc|^2 \geq \Delta R + \frac{2}{n}R^2.$$

One often use this evolution equation to get the finite blow up for Ricci flow the positive curvature. We shall consider the flow of warped product metrics and use this equation to get a lower bound which forces the metrics  $g(t)$  going to a flat metric. In many physical important examples of Einstein metrics, we meet the warped product metrics. So it is nature to consider the Ricci flow for warped product metrics.

Let  $M = \mathbb{R}_+ \times N^n$  with the product metric

$$g = \phi(x)^2 dx^2 + \psi(x)^2 \hat{g}.$$

Here  $(N^n, \hat{g})$  is an Einstein manifold of dimension  $n \geq 2$ ,  $\phi(x)$  and  $\psi(x)$  are two smooth positive functions of the variable  $x > 0$ . In this paper, we take  $N = S^n$  be the standard n-sphere so that  $\hat{g} = d\sigma^2$ . If  $n = 1$ , the related results on Ricci flow can be found in [8].

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Introduce the arc-length parameter  $s$  and sectional curvatures in the following way. Let

$$(1.3) \quad s = \int_0^x \phi(\tau) d\tau.$$

Let

$$K_0 = -\frac{\psi_{ss}}{\psi}.$$

be the sectional curvature of the 2-planes perpendicular to the spheres  $\{x\} \times S^n$  and

$$K_1 = \frac{1 - \psi_s^2}{\psi^2}$$

the sectional curvature of the 2-planes tangential to the spheres. Note that

$$\frac{\partial}{\partial s} = \frac{1}{\phi(x)} \frac{\partial}{\partial x}.$$

In terms of the arc-length parameter  $s$ , the metric can be read as

$$g = ds^2 + \psi^2 \hat{g}.$$

Recall that the standard hyperbolic  $n$ -space can be written as

$$(\mathbb{R}_+ \times S^{n-1}, dx^2 + \sinh^2 x \hat{g}).$$

Using this model, we give the following definition.

**Definition 1.** *We say the product metric  $g$  on the manifold  $M$  is of hyperbolic type if it has bounded negative sectional curvature with*

$$\frac{\phi(x)}{x} \approx 1,$$

and

$$\frac{\psi(x)}{\sinh(x)} \approx 1,$$

as  $x \rightarrow +\infty$ , where  $f \approx 1$  means that there exist two positive constants  $C_1$  and  $C_2$  such that  $C_1 \leq f \leq C_2$  uniformly for  $x$  large. We write by  $HM(n+1)$  the class of product metrics of hyperbolic type on  $M$ .

We remark that if we impose the conditions

$$\partial_x^\alpha(\phi(x)/x) \rightarrow 0, \quad \partial_x^\alpha\left(\frac{\psi(x)}{\sinh(x)}\right) \rightarrow 0,$$

for  $x \rightarrow +\infty$ , where  $1 \leq \alpha \leq 2$ , then we can have that

$$K_0 \rightarrow -1, \quad K_1 \rightarrow -1$$

as  $|x| \rightarrow \infty$ .

It is known that the Ricci curvature tensor of the product metric  $g$  is given by

$$Rc = n\left\{-\frac{\psi_{xx}\psi + \psi_x\phi_x}{\psi\phi}\right\}dx^2 + \left\{-\frac{\phi\psi_x\psi_{xx} - (n-1)\phi\psi_x^2 + \psi\phi_x\psi_x}{\phi^3} + n-1\right\}\hat{g}.$$

Let

$$g_s = \psi^2 \hat{g}.$$

Then we have the hessian formula for  $s$  (see [11]):

$$Hess(s) = \psi \psi_s \hat{g} = \frac{\psi_s}{\psi} g_s,$$

and the Ricci tensor can be written as

$$Rc = nK_0 ds^2 + [K_0 + (n-1)K_1] \psi^2 \hat{g}.$$

See [12] and [1].

Note that the scalar curvature is given by

$$R = nK_0 + n[K_0 + (n-1)K_1].$$

Then for the warped product metrics, the Ricci flow (1.1) becomes the system

$$(1.4) \quad \begin{cases} \psi_t = -[K_0 + (n-1)K_1]\psi = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi}, \\ \phi_t = -nK_0\phi = n\frac{\psi_{ss}}{\psi}\phi. \end{cases}$$

In the  $xt$  coordinates, we have

$$(1.5) \quad \begin{cases} \psi_t = \frac{1}{\phi(x)} \frac{\partial}{\partial x} \frac{1}{\phi(x)} \frac{\partial}{\partial x} \psi - (n-1) \frac{1 - (\frac{1}{\phi(x)} \frac{\partial}{\partial x} \psi)^2}{\psi}, \\ \phi_t = n \frac{\frac{\partial}{\partial x} \frac{1}{\phi(x)} \frac{\partial}{\partial x} \psi}{\psi}. \end{cases}$$

We prove the following:

**Theorem 2.** *Let  $n \geq 2$ . Given a complete initial metric  $g_0 \in HM(n+1)$ . Then the Ricci flow exists globally and keeps in the class  $HM(n+1)$ ; furthermore, as  $t \rightarrow \infty$ , the flow converges uniformly to the flat metric.*

We note that the local existence of Ricci flow has been proved by W. Shi [14] and the uniqueness result can be proved in the same way as in Theorem 3.3 in M. Simon [12] by using the harmonic map heat flow associated to the Ricci flow (see R. Hamilton [7]). We remark that in our case, the condition (4) in [12] has not been satisfied. However, the argument there can be modified to our hyperbolic case. The goal in the papers [12] and [1] (see also [4]) is to show that the pinching singularity in finite time along the Ricci flow occurs for some class of warped product metrics with non-negative curvature, as one can expect from the intuition. Although it is conjectured by R. Hamilton that the globally Ricci flow exists for the negative sectional curvature case, the convergence of the Ricci flow can not be easily classified.

We now give some remark about the possible application of our result to study the rigidity problem of hyperbolic warped product metrics. It may also be useful for our result to study the evolution of the quasi-local mass of Brown and York on the spheres (when  $n = 2$ ). To introduce the Brown-York mass of geodesic spheres, we look at the evolution of their mean curvatures.

For a fixed  $r > 0$ , let

$$\Sigma := \partial B(r) = \{p \in M; s(p) = r\}$$

be the sphere of radius  $r$  in the warped Riemannian manifold  $(M, g)$ . Note that the volume of  $\partial B(r)$  is

$$V(r) = |\Sigma| = \int_{\Sigma} d\sigma = \int_N \psi(s)^n dv_{\hat{g}} = \psi(r)^n V(N).$$

Here  $V(N)$  is the volume of  $N = S^n$  in the metric  $\hat{g}$ .

By the hessian formula for the function  $s$ , we have the mean curvature of the sphere  $\Sigma$  given by

$$(1.6) \quad H = \Delta s = n \frac{\psi_s}{\psi} = \frac{V'(s)}{V(s)}.$$

Let  $n = 2$ . Let's recall the definition of the Brown-York quasi-local mass (see [3] for definition and see [15] for more)

$$m(r) := m(\Sigma) = \int_{\Sigma} (H_0 - H).$$

where  $H_0$  is the mean curvature of  $\Sigma$  in the standard hyperbolic space. Recall that

$$H_0(x) = \frac{n \cosh(x)}{\sinh(x)}.$$

In our case, it is clear that

$$m(r) := m(\Sigma) = (H_0 - H)V(s) = H_0 V(r) - V_s(r).$$

Motivated by results in [10], [5], and [6], it is possible to prove some (weighted) monotonicity formula for the mass  $m(r)$  along the Ricci flow.

## 2. BASIC EVOLUTION EQUATIONS

In this section, we derive the evolution equations for the arc-length parameter  $s = s(x, t)$ , the volume function  $V$ , the curvature function  $K_0$ , and mean curvature function  $H$  along the Ricci flow (1.4). Some of these evolution equations for the derivatives of  $\psi$  were derived in [1]. We derive more formulae than what we can use in this paper since they may be useful to study the evolutions of quasi-local masses and ADM mass (see [3], [6], and [15] for the definition) along the Ricci flow.

Recall that the product metric  $g(t)$  is given by

$$g(t) = \phi(x, t)^2 dx^2 + \psi(x, t)^2 \hat{g},$$

which is

$$g(t) = ds^2 + \psi(s, t)^2 \hat{g}$$

in the st-coordinates. For a given smooth radial function  $f = f(s)$ , we have

$$\Delta f = \Delta_{g(t)} f = \frac{1}{\phi^2} f_{xx} + \frac{1}{\phi \psi^n} \left( \frac{\psi^n}{\psi} \right)_x f_x$$

or

$$\Delta f = f_{ss} + n \frac{\psi_s}{\psi} f_s.$$

First of all, along the Ricci flow, we have the following commutator relation (see [9]):

$$(2.1) \quad \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = \frac{\partial}{\partial t} \frac{\partial}{\partial s} - \frac{\partial}{\partial s} \frac{\partial}{\partial t} = -\frac{1}{\phi} \frac{\partial \phi}{\partial t} \frac{\partial}{\partial s} = -n \frac{\psi_{ss}}{\psi} \frac{\partial}{\partial s} = n K_0 \frac{\partial}{\partial s}.$$

We then consider the evolution equation for the arc-length parameters of the flow. By the equation (1.3), we have

$$\frac{\partial s}{\partial t} = \int^x \phi_t d\xi = n \int^s \frac{\psi_{ss}}{\psi} d\tau.$$

By this, we have

$$\frac{\partial s}{\partial t} = n \frac{\psi_s}{\psi} + n \int^s \frac{\psi_s^2}{\psi^2}.$$

Using the expression (1.6) for the mean curvature  $H$ , we have

$$(2.2) \quad \frac{\partial s}{\partial t} = H + \frac{1}{n} \int^s H^2 ds = \Delta s + \frac{1}{n} \int^s H^2 d\tau.$$

Note that under the Ricci flow (1.4), the volume  $V(t)$  is changed by

$$V_t = n \int_N \psi^{n-1} \psi_t dv_{\hat{g}} = n V(N) \psi^{n-1} (\psi_{ss} - (n-1) \frac{1 - \psi_s^2}{\psi}).$$

Then we can simplify it to the following form:

$$(2.3) \quad V_t = V_{ss} - \frac{n-1}{\psi^2} V = V_{ss} - (n-1) V(N)^{2/n} V^{(n-2)/n},$$

which is a parabolic equation for the volume  $V(s, t)$  of the spheres  $\Sigma$ . This gives us that

$$(\partial_t - \Delta)V = -\frac{n\psi_s}{\psi} V_s - (n-1) V(N)^{2/n} V^{(n-2)/n}.$$

Note that, when  $n = 2$ , we have

$$V_t = V_{ss} - V(N).$$

Using the Ricci flow equation (1.4), we have

$$(\partial_t - \Delta)\psi = K_1 \psi - \frac{n}{\psi}.$$

Using the commutator relation (2.1), we have

$$\partial_t \psi_s = \partial_s \psi_t - \frac{1}{\phi} \phi_t \psi_s.$$

Then we have

$$(2.4) \quad \partial_t \psi_s - (\psi_s)_{ss} = (n-2) \frac{\psi_{ss} \psi_s}{\psi} + (n-1) \frac{1 - \psi_s^2}{\psi^2} \psi_s,$$

which can also be written as

$$\partial_t \psi_s - (\psi_s)_{ss} = [-(n-2)K_0 + (n-1)K_1]\psi_s.$$

Here we have used that

$$(K_0)_s = -\frac{\psi_{sss}}{\psi} - K_0 \frac{\psi_s}{\psi}, \quad (K_1)_s = -\frac{\psi_s}{\psi}(K_1 - K_0).$$

Hence, by (2.4), we have

$$(2.5) \quad \partial_t \psi_s - \Delta(\psi_s) = [2K_0 + (n-1)K_1]\psi_s.$$

Let  $w = \psi_{ss}$ . Then we have

$$(2.6) \quad w_t - w_{ss} = (n-2)\frac{\psi_s}{\psi}w_s - (2K_0 - (4n-5)\frac{\psi_s^2}{\psi^2} + \frac{n-1}{\psi^2})w - 2(n-1)K_1\frac{\psi_s^2}{\psi}.$$

Hence,

$$(\partial_t - \Delta)w = -2\frac{n\psi_s}{\psi}w_s - (2K_0 - (4n-5)\frac{\psi_s^2}{\psi^2} + \frac{n-1}{\psi^2})w - 2(n-1)K_1\frac{\psi_s^2}{\psi}.$$

We now compute the evolution equation for the curvature function

$$K = -K_0 = \psi_{ss}/\psi,$$

which is more important to us. Using (2.6), we can derive that

$$(2.7) \quad K_t - K_{ss} = n\frac{n\psi_s}{\psi}K_s - 2K^2 - 4(n-1)K_1K - 2(n-1)K_1\frac{\psi_s^2}{\psi^2}.$$

That is to say,

$$K_t - \Delta K = -4K^2 - 2(n-1)K_1K - 2(n-1)K_1\frac{\psi_s^2}{\psi^2}.$$

We note that

$$H_t = -n\psi_s\frac{\psi_t}{\psi^2} + n\frac{\partial_t\partial_s\psi}{\psi} = -n\psi_s\frac{\psi_t}{\psi^2} + n\frac{1}{\psi}(\partial_s\psi_t - \frac{1}{\phi}\phi_t\psi_s).$$

Using the relations that

$$\frac{\psi_{ss}}{\psi} = \frac{H_s}{n} + \frac{H^2}{n^2}$$

and

$$\frac{\psi_{sss}}{\psi} = \frac{H_{ss}}{n} + \frac{3\psi_s\psi_{ss}}{\psi^2} - \frac{2H^3}{n^3} = \frac{H_{ss}}{n} + \frac{3HH_s}{n^2} + \frac{H^3}{n^3},$$

we can easily find that

$$(2.8) \quad H_t = H_{ss} + \frac{2n-1}{n}HH_s - \frac{1}{n^2}H^3 + 2(n-1)V(N)^{2/n}V^{-2/n}H.$$

Then we have

$$(\partial_t - \Delta)H = \frac{n-1}{n}HH_s - \frac{1}{n^2}H^3 + 2(n-1)V(N)^{2/n}V^{-2/n}H.$$

Let

$$a(x, t) = \psi^2(K_1 - K_0)$$

be the measure of the difference of the two kinds of sectional curvatures. Then we have

$$a_t = a_{ss} + (n-4)\frac{\psi_s}{\psi}a_s - 4(n-1)\frac{\psi_s^2}{\psi^2}a,$$

which is

$$(\partial_t - \Delta)a = -4\frac{\psi_s}{\psi}a_s - 4(n-1)\frac{\psi_s^2}{\psi^2}a.$$

We can use the maximum principle to this equation to find a pinching estimate.

### 3. PROOF OF THEOREM 2

In this section we mainly prove Theorem 2.

For any given smooth invertible function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we consider the evolution equations for the function  $u := f(\psi)$ . Note that

$$u_t = f'\psi_t, \quad u_s = f'\psi_s,$$

and

$$u_{ss} = f'\psi_{ss} + f''\psi_s^2.$$

Then

$$u_t - u_{ss} = f'(\psi_t - \psi_{ss}) - \frac{f'' \circ f^{-1}}{|f' \circ f^{-1}|^2}u_s^2 = -f'((n-1)K_1 + n)\psi - \frac{f'' \circ f^{-1}}{|f' \circ f^{-1}|^2}u_s^2,$$

and

$$(3.1) \quad u_t - u_{ss} = -((n-1)K_1 + n)f' \circ f^{-1}(u)f^{-1}(u) - \frac{f'' \circ f^{-1}}{|f' \circ f^{-1}|^2}u_s^2.$$

We shall use this evolution equation to study the behavior of the function  $\psi$ . In particular, for  $u = f(u) = \psi^k$  for  $\psi < 1$  and  $u = 1$  for  $\psi \geq 1$ , we have that  $f^{-1} = u^{1/k}$ ,  $f' = k\psi^{k-1}$ ,  $f'' = k(k-1)\psi^{k-2}$ , and

$$u_t - u_{ss} = -k((n-1)K_0 + n)u - \frac{k-1}{k}u^{-1}u_s^2.$$

If  $k$  is a negative integer, we know that from the argument as in [6] that  $\psi(x, t)$  has the same decay as  $\psi(x, 0)$  at infinity.

We now need to study the behavior of functions  $\psi$  and  $\phi$  at  $x = 0$ . From the definition of Riemannian metric, we know that for all  $t > 0$ ,

$$s(0, t) = 0, \quad s_x(0, t) = 0 = \phi(0, t), \quad \phi_x(0, t) = 1 \quad \psi(0, t) = 0, \quad \psi_s(0, t) = 1.$$

Using the formulae for curvature  $K_0$  and bounded curvature conditions we have

$$\psi_{ss}(0, t) = 0.$$

Using the maximum principle (see [14]) to the evolution equation (2.4), we know that

$$\psi_s > 1, \quad \text{for } x > 0, t > 0.$$

This implies that  $K_1 < 0$  for  $x > 0$  and  $t > 0$ .

Using the maximum principle to the evolution equation (2.6), we know that

$$\psi_{ss} > 0, \text{ for } x > 0, t > 0,$$

which gives us that  $K_0 < 0$  for  $x > 0$  and  $t > 0$ . Then we have  $R(t) < 0$  for all  $t > 0$ .

Taking s-derivative in the formulae  $\psi_{ss} = -K_0\psi$ , we get that  $\psi_{sss}(0, t) = -K_0(0, t)\psi_s(0, t) \geq 0$ .

Using the maximum principle to the evolution equation (1.5), we know that

$$\psi > 0, \quad \phi > 0$$

for all  $t > 0$  and  $x > 0$ .

Using the negativity of the curvatures  $K_0$  and  $K_1$ , the scalar curvature evolution equation (1.2), and the maximum principle, we have the following uniform lower bound of the scalar curvature  $R$ ,

$$(3.2) \quad 0 > R(t) \geq \frac{-1}{2t/n - (\inf_M R(0))^{-1}}.$$

Hence we have the uniform bound for the sectional curvature functions  $K_0$  and  $K_1$ . Using Shi's estimate (see Theorem 1.1 in [14]), we know that the flow  $g(t)$  exists globally, and by our estimate (3.2), as  $t \rightarrow \infty$ , the flow  $g(t)$  converges to a smooth metric  $g_\infty$  with the zero sectional curvature.

We now consider the behavior of the solution pair  $(\phi, \psi)$  to the flow (1.4). By (1.4), we have that

$$\partial_t(\log \psi) = -[K_0 + (n-1)K_1], \quad \partial_t(\log \phi) = -nK_0.$$

Integrating over the time interval  $[0, t]$ , we get

$$\psi(x, t) = \psi(x, 0) \exp\left(-\int_0^t [K_0 + (n-1)K_1]\right),$$

and

$$\phi(x, t) = \phi(x, 0) \exp\left(-\int_0^t K_0\right),$$

which say that the behavior of  $\psi(x, t)$  and  $\phi(x, t)$  are equivalent to  $\psi(x, 0)$  and  $\phi(x, 0)$  respectively at infinity  $|x| \rightarrow \infty$ .

This proves that the Ricci flow keeps the class  $HM(n+1)$ .

By this, we complete the proof of Theorem 2.

#### 4. DISCUSSIONS

Looking at the example of the flow  $(g(t))$  defined by

$$\partial_t g(t) = 2g(t).$$

Then we have

$$g(t) = e^{2t}g(0).$$

Hence, as  $t \rightarrow +\infty$ , we have

$$g(t) \rightarrow g_\infty,$$



where  $g_\infty$  is a flat metric. In this sense, Theorem 2 is not surprising.

To make the limit metric of the Ricci flow to have negative curvature, we may study the following modified Ricci flow:

$$(4.1) \quad \partial_t g(t) = -2Rc(g(t)) - 2ng, \quad t \in I$$

If  $g(t)$  is a modified Ricci flow, then the family defined by

$$\bar{g}(\tau) = (1 + n\tau)g\left(\frac{1}{n} \log(1 + n\tau)\right)$$

is a Ricci flow. Hence, we can use our Theorem 2 to get globally existence of the modified Ricci flow (4.1). However, we shall leave the convergence topic of the global modified Ricci flow to future study.

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